

Closing Tues: HW 12.4, 13.2

Exam 2 is next Thursday!

Covers:

10.1-10.3: Analyzing a function

11.1,2: Deriv. of $\ln(x)$ and e^x

12.1,3,4: Integrals, finding C

13.2: Definite Integrals

Entry Task (from HW):

$$AC'(q) = -\frac{4}{q^2} + \frac{1}{4}$$

$$AC(4) = 10$$

Find $AC(q)$, $TC(q)$, and FC .

$$\begin{aligned} AC(q) &= \int -4q^{-2} + \frac{1}{4} dq \\ &= -\frac{4}{-1}q^{-1} + \frac{1}{4}q + C \end{aligned}$$

$$AC(q) = 4\frac{1}{q} + \frac{1}{4}q + C$$

$$\begin{aligned} AC(4) = 10 &\Rightarrow 4 \cdot \frac{1}{4} + \frac{1}{4}(4) + C = 10 \\ &\quad \underbrace{2 + C = 10} \\ &\quad C = 8 \end{aligned}$$

Total Values and Marginal Values	Total Values and Average Values
$TR(x) = \int MR(x)dx$ $TR'(x) = MR(x)$	$AR(x) = \frac{TR(x)}{x}$ $TR(x) = xAR(x)$
$TC(x) = \int MC(x)dx$ $TC'(x) = MC(x)$	$AC(x) = \frac{TC(x)}{x}$ $TC(x) = xAC(x)$
$P(x) = \int MP(x)dx$ $P'(x) = MP(x)$	

Initial conditions: $TR(0) = 0, TC(0) = FC$

Note: $P(x) = TR(x) - TC(x)$

$$\begin{aligned} AC(q) &= \frac{4}{q} + \frac{1}{4}q + 8 \\ TC(q) &= 4 + \frac{1}{4}q^2 + 8q \\ FC &= TC(0) = 4 \end{aligned}$$

Section 13.2: Definite integrals and the Fundamental Theorem of Calculus

Recall: An *indefinite integral* is a function (the general antiderivative)

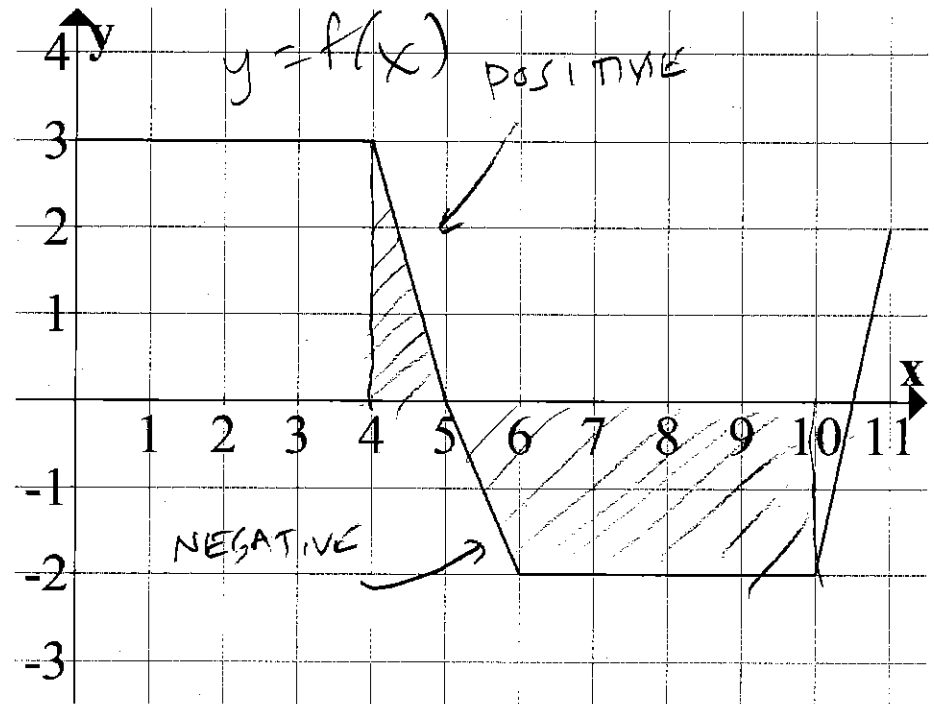
$$\int f(x) dx = F(x) + C$$

New: A *definite integral* is a number that represents *net area*

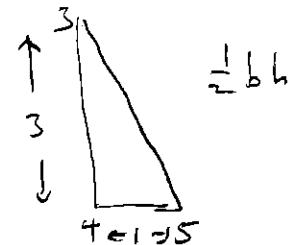
$$\int_a^b f(x) dx = \text{"net area between } f(x) \text{ and the } x\text{-axis from } x = a \text{ to } x = b\text{"}$$

Notes

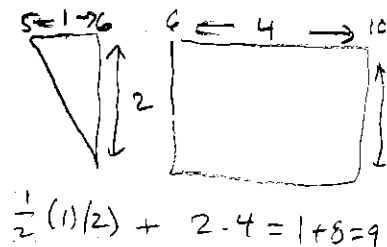
Above the x-axis counts as positive area.
Below the x-axis counts as negative area.
"a" and "b" are called the *bounds*, or *limits*, of integration.



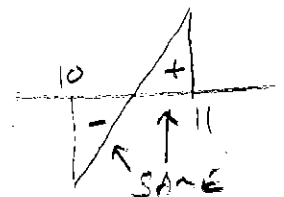
$$\int_4^5 f(x) dx = \frac{1}{2} (1) (3) = \frac{3}{2} = 1.5$$



$$\int_5^{10} f(x) dx = -9$$



$$\int_{10}^{11} f(x) dx = 0$$



Now consider

$$A(m) = \int_0^m f(x) dx$$

= "accumulated net area from 0 to m"

Using the same graph, what is

$$A(0) = \int_0^0 f(x) dx = 0$$

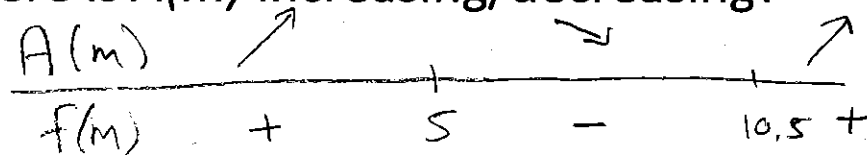
$$A(4) = \int_0^4 f(x) dx = 3 \cdot 4 = 12$$

$$A(5) = \int_0^5 f(x) dx = 12 + 1.5 = 13.5$$

$$A(8) = \int_0^8 f(x) dx = 13.5 - \frac{1}{2}(1)(2) = 12.5$$
$$= 13.5 - 1 - 4 = 8.5$$

Questions/Observations:

Where is $A(m)$ increasing/decreasing?



$A(m)$ increases (adds positive area) for $0 \leq m < 5$ and for $m > 10.5$.

NOTE: $m=5$ is a local max
 $m=10.5$ is a local min.

See any connections for $A(m)$ and $f(x)$?

$$A'(m) = f(m)$$

$A(m)$ = an antiderivative of $f(m)$

What does $A(5) - A(4)$ represent?

$$A(5) - A(4) = \int_4^5 f(x) dx = 1.5$$

In addition, in the activities you found:

1. "the area under the speed graph" equals "the change in distance".

$$\int_a^b s(t) dt = D(b) - D(a)$$

2. "the area under the MR/MC graph" equals "the change in TR/TC"

$$\int_a^b MR(x) dx = TR(b) - TR(a)$$

$$\int_a^b MC(x) dx = TC(b) - TC(a)$$

These are examples of a profound fact about anti-derivatives and areas.

The Fundamental Theorem of Calculus

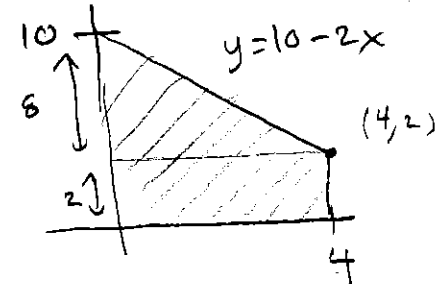
If $F(x)$ is any anti-derivative of $f(x)$, then

$$\int_a^b f(t) dt = F(b) - F(a)$$

Example: Find the area under

$$MR(x) = 10 - 2x$$

from $x = 0$ to $x = 4$.



2 options

I GEOMETRY

$$AREA = 2 \cdot 4 + \frac{1}{2} (4)(8) = 8 + 16 = \boxed{24}$$

II USE FTC.

$$\int_0^4 10 - 2x dx$$

$$= 10x - x^2 \Big|_0^4$$

$$= \underbrace{(10(4) - (4)^2)}_{F(4)} - \underbrace{(10(0) - (0)^2)}_{F(0)}$$

$$= \boxed{24}$$

$$F(x) = 10x - x^2$$

$$F(4) = 10(4) - (4)^2 = 24$$

$$F(0) = 10(0) - (0)^2 = 0$$

How to compute definite integrals

Step 1: Find *any* antiderivative, $F(x)$.

(usually we pick $C = 0$, but you use any C value you want and it doesn't change the answer)

Step 2: Evaluate $F(x)$ at $x = b$ and $x = a$.

Step 3: Subtract

We do all this in one line as follows:

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

More Examples:

$$1. \int_1^2 6x^2 - 2x + 5 dx$$

$$= 6 \frac{1}{3} x^3 - 2 \frac{1}{2} x^2 + 5x \Big|_1^2$$

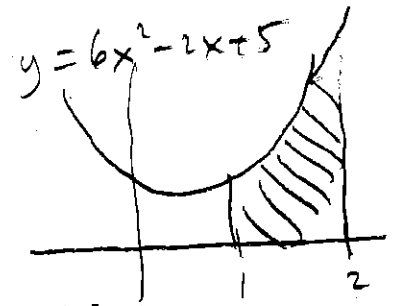
$$= 2x^3 - x^2 + 5x \Big|_1^2$$

$$= \underbrace{(2(2)^3 - (2)^2 + 5(2))}_{F(2)} - \underbrace{(2(1)^3 - (1)^2 + 5(1))}_{F(1)}$$

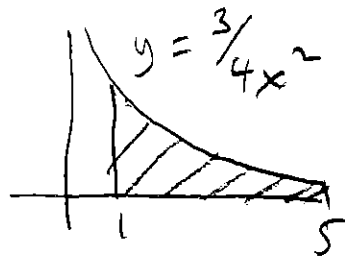
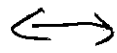
$$= (16 - 4 + 10) - (2 - 1 + 5)$$

$$= 22 - 6$$

$$= \boxed{16}$$



$$2. \int_1^5 \frac{3}{4x^2} dx$$



$$= \int_1^5 \frac{3}{4} x^{-2} dx$$

$$= \frac{3}{4} \frac{1}{-1} x^{-1} \Big|_1^5$$

$$= -\frac{3}{4} \frac{1}{x} \Big|_1^5$$

$$= \left(-\frac{3}{4} \frac{1}{(5)} \right) - \left(-\frac{3}{4} \frac{1}{(1)} \right)$$

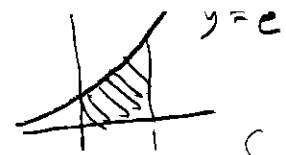
$$= -\frac{3}{20} + \frac{3}{4}$$

$$= -\frac{3}{20} + \frac{15}{20}$$

$$= \frac{12}{20} = \boxed{\frac{3}{5}}$$

$\times \frac{5}{5}$
common denominator

$$3. \int_0^1 e^{x/3} dx$$



$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

$$= \int_0^1 e^{\frac{1}{3}x} dx$$

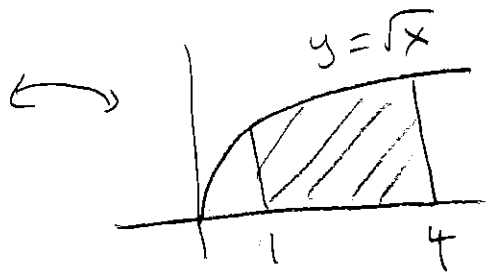
$$= \frac{1}{1/3} e^{\frac{1}{3}x} \Big|_0^1$$

$$= 3 e^{\frac{1}{3}x} \Big|_0^1$$

$$= \left(3 e^{\frac{1}{3}(1)} \right) - \left(3 e^{\frac{1}{3}(0)} \right)$$

$$= \boxed{3 e^{1/3} - 3 \approx 1.1868}$$

$$4. \int_1^4 \sqrt{x} \, dx$$



$$\int_1^4 x^{1/2} \, dx$$

$$= \frac{1}{3/2} x^{3/2} \Big|_1^4$$

$$= \frac{2}{3} x^{3/2} \Big|_1^4$$

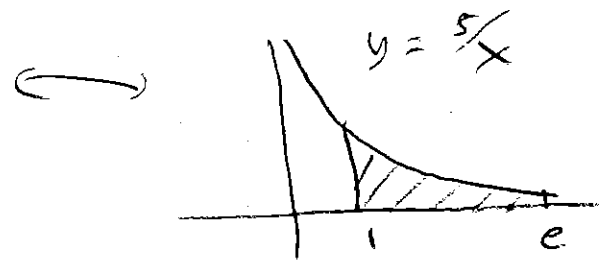
$$= \left(\frac{2}{3} (4)^{3/2} \right) - \left(\frac{2}{3} (1)^{3/2} \right)$$

$$= \frac{2}{3} \cdot 8 - \frac{2}{3}$$

$$= \frac{16}{3} - \frac{2}{3}$$

$$= \boxed{\frac{14}{3}}$$

$$5. \int_1^e \frac{5}{x} \, dx$$



$$\int_1^e 5 \cdot \frac{1}{x} \, dx$$

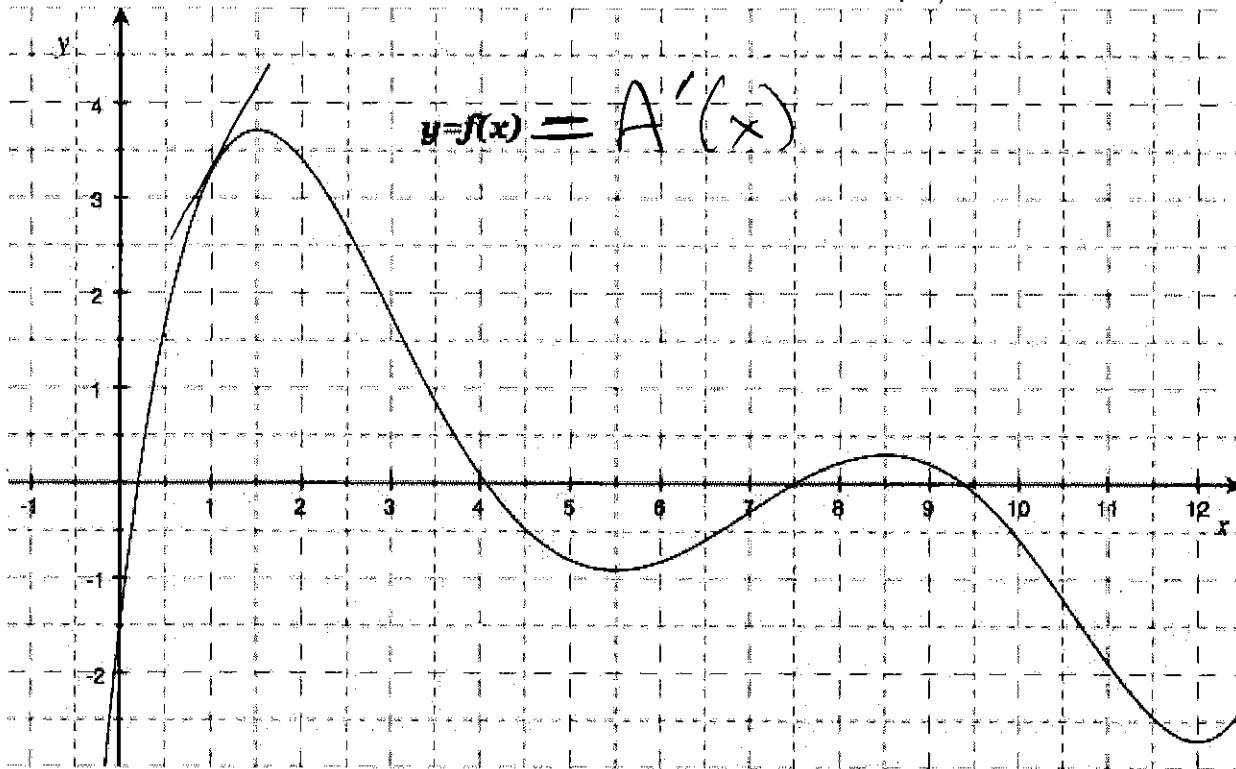
$$= 5 \ln(x) \Big|_1^e$$

$$= (5 \ln(e)) - (5 \ln(1))$$

$$= 5 - 0$$

$$= \boxed{5}$$

3. (18 points) Below is the graph of a function $y = f(x) = A'(x)$



Define the function $A(m)$ by $A(m) = \int_0^m f(x) dx$.

NOTE: You do **not** need to show any work for the problems on this page.

(a) Name all values of m at which $A(m)$ has a local minimum. $A \rightarrow \nearrow \searrow \nearrow \searrow$
 $A' = 0.25 + 4 - 7.5 + 9.5 -$

ANSWER: $m = 0.25, 7.5$

(b) Give the one-minute interval over which $A(m)$ increases the most. where A' is biggest

biggest slope of A

ANSWER: from 1 to 2

(c) True or False?

circle one

T F $A(2.51) > A(2.50)$

$A'(2.50)$ is positive $\Rightarrow A$ INCREASING

T F $f(2.51) > f(2.50)$

f is decreasing at 2.5

T F $A(10.01) > A(10.00)$

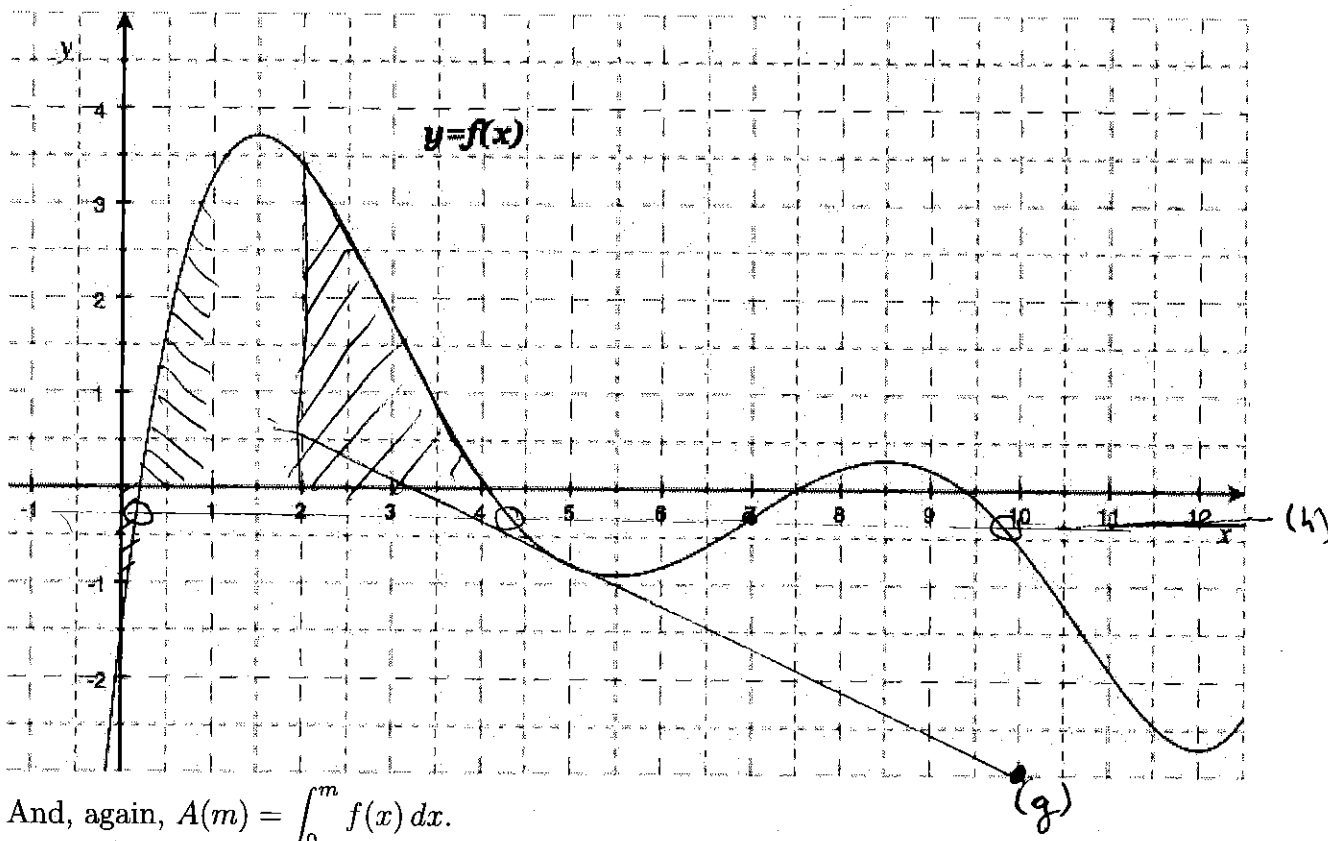
$A'(10)$ is negative $\Rightarrow A$ DECREASING

T F $f'(1.00) > f'(1.01)$

slope of f is decreasing at 1.

(THIS PROBLEM IS CONTINUED ON THE NEXT PAGE.)

Here is the graph of $y = f(x)$ again.



And, again, $A(m) = \int_0^m f(x) dx$.

NOTE: The problems on this page **require some justification**: clearly mark points and lines on the graph, shade areas, show calculations of slopes and areas, etc.

(e) Compute $A(1)$. $\nabla \quad \triangle$

$$\text{AREA} \approx -\frac{1}{2}(1.5)(0.25) + \frac{1}{2}(0.75)(3.5) \approx 1.125$$

ANSWER: $A(1) \approx \underline{1.125}$

(f) Compute $A'(12) = f(12) \approx -2.7$

ANSWER: $A'(12) \approx \underline{-2.7}$

(g) Compute $A''(5) = f'(5) = \text{slope at } 5$

Two points
on tangent line

$$\begin{aligned} &(10, -3) \\ &(5, -0.75) \end{aligned}$$

$$\frac{-0.75 - (-3)}{5 - 10} =$$

ANSWER: $A''(5) \approx \underline{-0.45}$

(h) Name a value of x at which $f(x) = f(7)$.

HEIGHT
AT 7

ANSWER: $x = \underline{0.2, \text{ or } 4.25, \text{ or } 9.75}$

(i) Compute $A(4) - A(2)$.

$$A(4) - A(2) = \int_2^4 f(x) dx = \frac{1}{2}(2)(3.5)$$

= area from 2 to 4
ANSWER: $A(4) - A(2) = \underline{\approx 3.5}$